Physics 606 Exam 1

Please be well-organized, and show all significant steps clearly in all problems.

You are graded on your <u>work</u>.

An answer, even if correct, will receive zero credit unless it is obtained via the work shown.

Do all your work on blank sheets, and turn them in as scanned files, writing your name clearly.

It is implicit that you pledge to communicate with no one during the exam (except for questions to me about the meaning of the problems).

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \qquad\qquad \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$$

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1. For the general state $|n\rangle$ of a 1-dimensional harmonic oscillator ($n = 0,1,2,\ldots$):

(a) (10) Calculate the uncertainty in position Δx.

(b) (10) Calculate the uncertainty in momentum Δp, and the product $\Delta p\ \Delta x$.

[In all problems you are graded on your well-organized, clear, and complete work, not the answer.]

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2. (a) (10) Calculate the Fourier transform of the 3-dimensional Dirac delta function $\delta^{(3)}\left(\vec{r}-\vec{r}_0\right)$.

(This is the 3d generalization of $\delta\left(x-x_0\right)$.)

(b) (10) Write $\delta^{(3)}\left(\vec{x}\right)$ in terms of its Fourier transform, in a form like

$$\delta^{(3)}\left(\vec{r}-\vec{r}_0\right) = \int d^3k\ \mathrm{e}^{i\vec{k}\cdot\vec{r}}\ \text{(something)} \quad .$$

3. Consider the following two **time-dependent** operators for a 1-dimensional harmonic oscillator in the **Schrödinger picture**:

$$X(t) = x\cos(\omega t) - \frac{1}{m\omega} p\sin(\omega t)$$
$$P(t) = p\cos(\omega t) + m\omega\, x\sin(\omega t) \quad .$$

Now let us transform to the Heisenberg picture, where $x(t)$ and $p(t)$ are the Heisenberg position and momentum operators. Let $X_H(t)$ and $P_H(t)$ be the Heisenberg operators corresponding to $X(t)$ and $P(t)$. Recall that the Heisenberg equation of motion is

$$\frac{dA_H(t)}{dt} = \frac{1}{i\hbar}\left[A_H(t), H(t)\right] + \frac{\partial A_H(t)}{\partial t}$$

for an operator $A(t)$ which already has a time dependence in the Schrödinger picture.

(a) (10) Calculate the commutator $\left[X_H(t), P_H(t)\right]$.

(b) (10) Calculate $\frac{dX_H(t)}{dt}$ using the Heisenberg equation of motion and the Hamiltonian for the harmonic oscillator.

(c) (10) Calculate $\frac{dP_H(t)}{dt}$ (again using the Heisenberg equation of motion and the Hamiltonian for the harmonic oscillator).

4. We made order-of-magnitude estimates based on the uncertainty principle, where the estimated energy can lie either above or below the true value. Better estimates are provided by the variational method, where the energy for an approximate trial wavefunction ψ (with the exact Hamiltonian) is guaranteed not to lie below the true ground state energy E_0: the variational theorem states that

$$\frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \geq E_0 \quad .$$

If one solves the Schrödinger equation for a *finite square well* in 1 dimension, it is found that there is always a bound state. Here let us do better by proving that for *any attractive potential*, with

$$V(x) \leq 0 \text{ for all } x \quad , \quad V(x) < 0 \text{ for some range of } x \, ,$$

there is always a bound state, with

$$E_0 < 0$$

where E_0 is the expectation value of the energy for this state with the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) \ .$$

As a trial wavefunction, let us use the normalized Gaussian

$$\psi(x) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2} \ .$$

To give the proof, you need to do the following:

(a) (10) Evaluate the expectation value

$$E(\alpha) = \int_{-\infty}^{\infty} dx \, \psi^* H \psi$$

of the Hamiltonian H in the state ψ as a function of the variational parameter α and an integral involving $V(x)$.

(b) (10) Minimize $E(\alpha)$ with respect to α to find the best variational state, and thus the optimum value of α, in terms of integrals involving $V(x)$.

(c) (10) Put this best value of α into the expression for $E(\alpha)$ and show that it is negative:

$$E(\alpha) < 0 \quad \text{for the state with the optimized value of } \alpha \, .$$

Be well-organized, clear, and complete in your work.

The true ground-state wavefunction in this arbitrary potential $V(x)$ cannot lie above this trial function, so it must also have negative energy.